# INVESTIGATIONS OF KENNELLY-HEAVISIDE LAYER HEIGHTS FOR FREQUENCIES BETWEEN 1,600 AND 8,650 KILOCYCLES PER SECOND [1]

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**ABSTRACT**

The results of observations of the height of the Kennelly-Heaviside layer carried out near Washington, D. C., during 1930 are presented. Evidence for the existence of two layers (corresponding closely in virtual height to the E and F regions discussed by Professor Appleton) is found during daylight on frequencies between three and five megacycles. The modification in the virtual height of the higher F .ayer produced by the existence of a low E layer is investigated theoretically, and the possibility of large changes in virtual height near the highest frequency returned by the E layer is pointed out. A number of oscillograms showing the characteristic types of records observed during the tests are presented together with a graph of average heights from January to October, 1930.

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## I. INTRODUCTION

It is the purpose of this paper to describe tests conducted by the National Bureau of Standards for the study of the Kennelly-Heaviside layer subsequent to those recently reported.[2] The method employed in these tests is the group retardation or pulse method described by Breit and Tuve.[3] In this method pulses of about $5 \times 10^{-4}$ second duration are transmitted with a group frequency of about 30 pulses per second. The signals received at a distant point are recorded by means of an oscillograph.

Fig. 1 (a) shows the rectified form of the pulse as received in the absence of sky waves. Received patterns showing more than one peak for a single transmitted pulse are attributed to waves arriving over more than one path; that is, a ground wave and single or multiple reflections from one or more ionized strata or layers. Patterns indicating reflections are shown in Figure 1 (b).

The pulse method has been extensively employed in America in Kennelly-Heaviside layer studies. In England, phase interference

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[1] Presented at annual meeting of American Section, International Scientific Radio Union, May 1, 1931.

[2] T. R. Gilliland, Kennelly-Heaviside Layer Height Observations for 4,045 kc. and 8,650 kc., B. S. Jour. Research, vol. 5, RP246, p. 1057; 1930, and Proc. I. R. E., vol. 19, p. 114; 1931. P. A. DeMars, T. R. Gilliland, G. W. Kenrick, Kennelly-Heaviside Layer Studies, Proc. I. R. E., vol. 19, pp. 106–113; January, 1931.

[3] G. Breit and M. A. Tuve, A Test of the Existence of a conducting Layer, Phys. Rev., vol. 21, p. 554; 1926; and Proc. I. R. E., vol. 17, p. 1513; 1929.

methods have been used by Prof. E. V. Appleton and coworkers.[4] It is of interest to note the good accord of the results obtained by the methods outlined here with those of Professor Appleton.

Both methods have peculiar advantages, and, as has been emphasized, depend upon somewhat different analytical relations for their interpretation. It has been shown, however, that the results obtained by both methods should be in accord.[5] A brief review and comparison of the methods follows; the reader is referred to the references noted for a more extended discussion.

Briefly, Appleton's method utilizes the phase reinforcements and cancellations produced by slight changes in emitted frequency due to the presence of multiple transmission paths having different phase retardations (that is, due to sky waves). The group retardation observations, on the other hand, measure the time retardation between the pulses arriving over different paths. The Appleton method employs balanced open antenna and loop antenna systems which permit measurements of polarizations and other phenomena not directly disclosed by group retardation studies when a nondirectional antenna is employed. However, directional antennæ may be used to advantage to obtain additional information when pulse methods are employed. An interesting preliminary application of pulse methods to a study of phase relations has been described.[6] During the observations described here it was found convenient to combine the input from a horizontal and a vertical antenna in suitable proportions, in order to render the amplitudes of the ground and sky waves comparable.

The group retardation method has the advantage that sky waves arriving from several paths of widely different retardations (or from several distinct layers) are readily and immediately resolved from the oscillograms, without the use of the more elaborate harmonic analysis required in the interpretation of results obtained by phase interference methods. (See fig. 1 (b).) This is of marked importance when multiple reflections and a large number of paths contribute to the results. (See fig. 5.) Investigations of this type have emphasized that, particularly during night conditions, the number of paths may be very great indeed and far from sharply defined.[7] Pulse methods may be used to investigate transmission on relatively low frequencies, down to perhaps 200 kc. per second. On lower frequencies, however, the length of the desired pulse begins to approach the period of the transmitted frequency and the time constant of the antenna. After this, difficulties introduced by transients and other obvious modulation limitations govern. The phase interference methods are, however, not free from difficulty at these lower frequencies, due to the large change of frequency required to produce the interference patterns. Under such conditions, a modification of the phase interference method, in which the path is varied instead of the frequency, is of

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[4] See, for instance, E. V. Appleton, Proc. Royal Soc., vol. 109, p. 621; 1925; vol. 113, p. 450; 1926; vol. 115, p. 305; 1927; vol. 117, p. 576; 1929; vol. 126, p. 542; 1930; vol. 128, p. 134; 1930, vol. 128, p. 158; 1930. An application of a method essentially equivalent to the pulse method of Breit and Tuve has been recently described by T. L. Eckersley, Multiple Signals in Short Wave Transmission, Proc. I. R. E.; January, 1930.

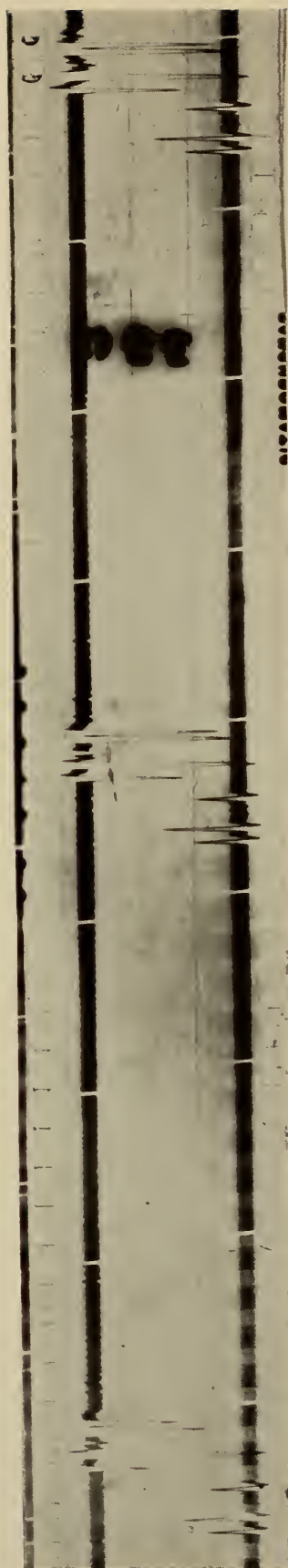[5] E. V. Appleton, Proc. Physical Soc. (London), vol. 41, p. 43, 1928; also vol. 42, p. 321; June, 1930. Kenrick and Jen, Proc. I. R. E., vol 17, pp. 711–733; April, 1929.

[6] L. R. Hafstad and M. A. Tuve, An Echo Interference Method for the Study of Radio Wave Paths, Proc. I. R. E., vol. 17, pp. 1788–1792; October, 1929.

[7] de Mars, Gilliland, and Kenrick, Proc. I. R. E., vol. 19, pp. 106–113; January, 1931.
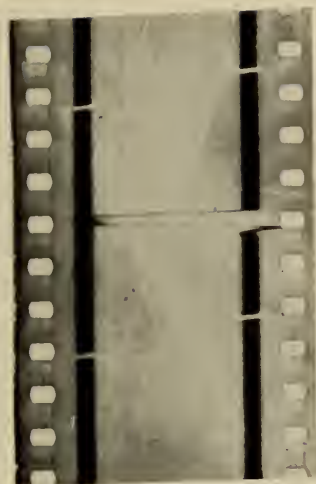
FIGURE 1.—*Types of oscillograms*

a, Type of pulse received in the absence of sky waves (interval between adjacent timing marks in all oscillograms shown is $\frac{1}{120}$ second); b, lower trace shows pulse pattern received at 11.20 a. m., E. S. T., on August 16, 1930 (frequency 4,015 kc). Note strong peaks corresponding to 100 km and 330 km virtual heights in addition to ground wave peak Upper trace shows more complicated pattern at 11.24 a .m. Here we find strong peaks corresponding to virtual heights of 100, 290, and 330 km. Both traces read from left to right
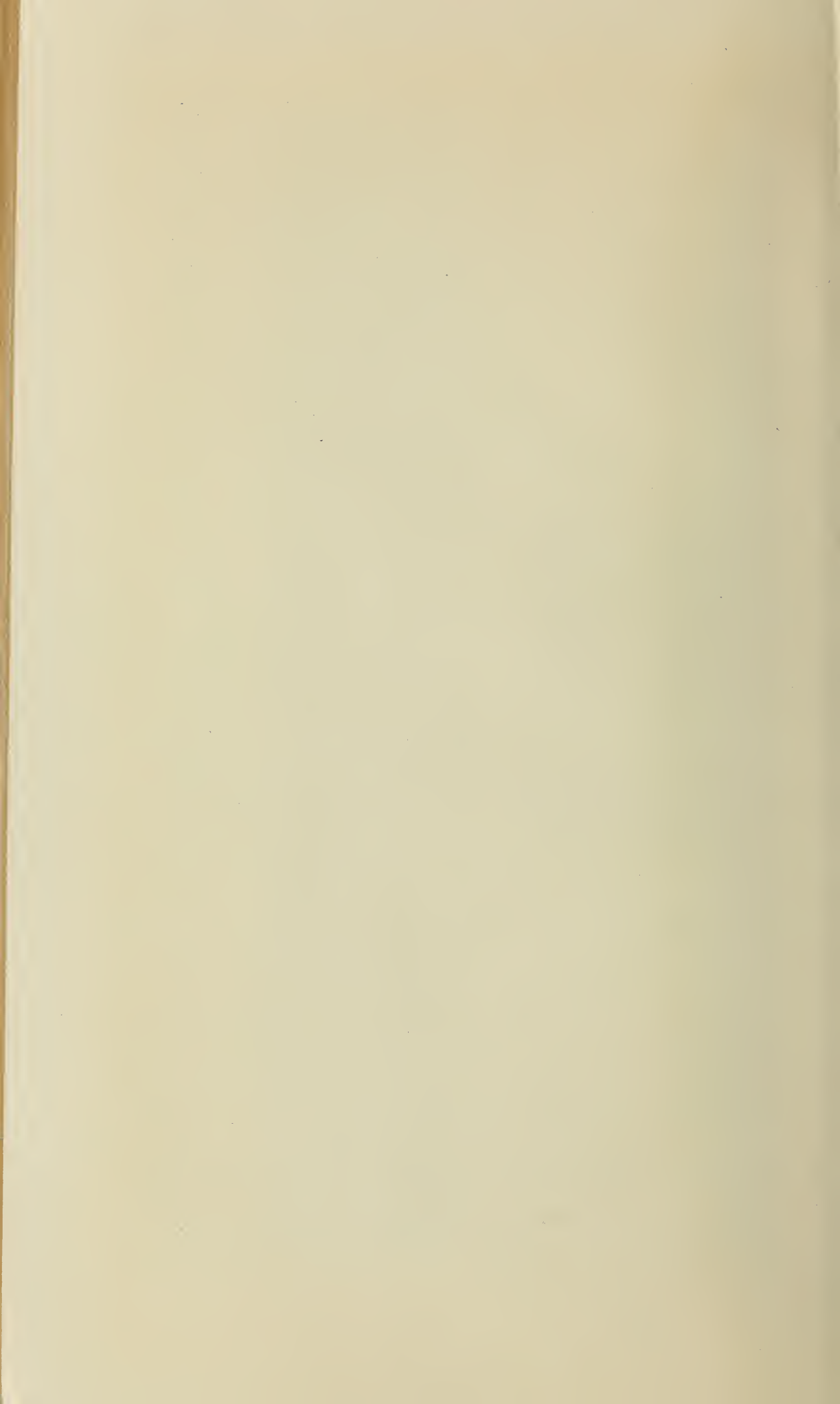
particular interest.[8]  The good accord obtained by the various methods, despite the difficulties peculiar to each, is encouraging.  Some comparisons of this sort are included here.

In the work described here oscillographic records were made on frequencies ranging from 1,600 kc per second to 8,650 kc per second (including a number not previously observed).

The observations include the summer of 1930, which was one of marked magnetic disturbance.  Observations during this period are hence of particular interest, although they can hardly be considered as typical summer conditions.  Of particluar interest in this connection is the notable absence of multiple peaks on the higher frequencies during the summer months, although they were frequently observed on 8,650 kc during the winter and early spring.  (See, for instance, the curves of Figure 2 where the average daytime virtual heights on 4,045 kc and 8,650 kc are indicated.)  It should be pointed out that although the same transmitter was used on all of the frequencies during August and September, the radiated power was not measured and may have been enough less on the highest frequencies to indicate a cut-off frequency somewhat lower than that actually existing.

It will be noted that these curves show only predominant virtual heights.  Diagrams are, however, also included, showing the observed virtual heights throughout the 24 hours on a number of frequencies. (See figs. 3, 4, 7, and 8.)  Distinct evidence of the existence of two sharply defined regions corresponding to the $E$ and $F$ regions of Appleton, is also to be noted in a number of cases.  An approximate analytical investigation of the effect of a lower layer on observed virtual heights from higher layers is also indicated.

## II. 4045-kc OBSERVATIONS

Figure 2 presents graphs giving average daylight virtual heights on 4,045 and 8,650 kc, from January, 1930 to October, 1930.  The observations from January to June were on transmission from the Naval Research Laboratory at Bellevue, Anacostia, D. C.  These observations were interrupted when no transmitting facilities were available at the Naval Research Laboratory for this purpose during the summer of 1930.  However, during the early summer a transmitter was placed in operation at the National Bureau of Standards experimental station near Alexandria, Va., which permitted the observations described to be made during August and September. All oscillograms of the received signals were recorded at the National Bureau of Standards experimental station at Kensington, Md.  The distance from the transmitter to receiver was, in each case, about 20 km.

It will be noted by reference to Figure 2 that on 8,650 kc only single pulses were received, except for the period from February 6 to April 7 when multiple patterns were obtained.  A rise in virtual height on 4,045 kc followed the disappearance of the multiple patterns on 8,650 kc.  It will be noted that this high virtual height persisted during the summer with temporary fluctuations to lower values, but in all

[8] J. Hollingworth, The Propagation of Radio Waves, J. I. E. E., (London) pp. 579–595; May, 1926. A recent interesting application of this method has also been reported on much higher frequencies.  See C. B. Mirick and E. R. Hentschel, Proc. I. R. E., vol. 17, pp. 1034–1041; June, 1929.  At the higher frequencies a rapidly moving recording system is essential to this method due to frequent inadvertent path changes during the observations from other causes, such as layer movements, etc.

cases the pattern on 8,650 kc remained single during the period of observations reported.

In addition to the very high values of the virtual height of the $F$ layer, distinct evidence of the existence of the $E$ layer on 4,045 kc during the daytime was to be found on numerous occasions during the middle of August. (Fig. 3.) Figure 1(*b*) gives an oscillograph trace showing this phenomenon, and scatter diagrams contrasting this effect with conditions a few days later are shown in Figure 4. The actual values of virtual heights as read from the records are plotted as points, the lowest distinct height from each record being indicated by a cross, and distinct greater heights (not in simple multiple relation) are denoted by dots. The marked frequency with which the
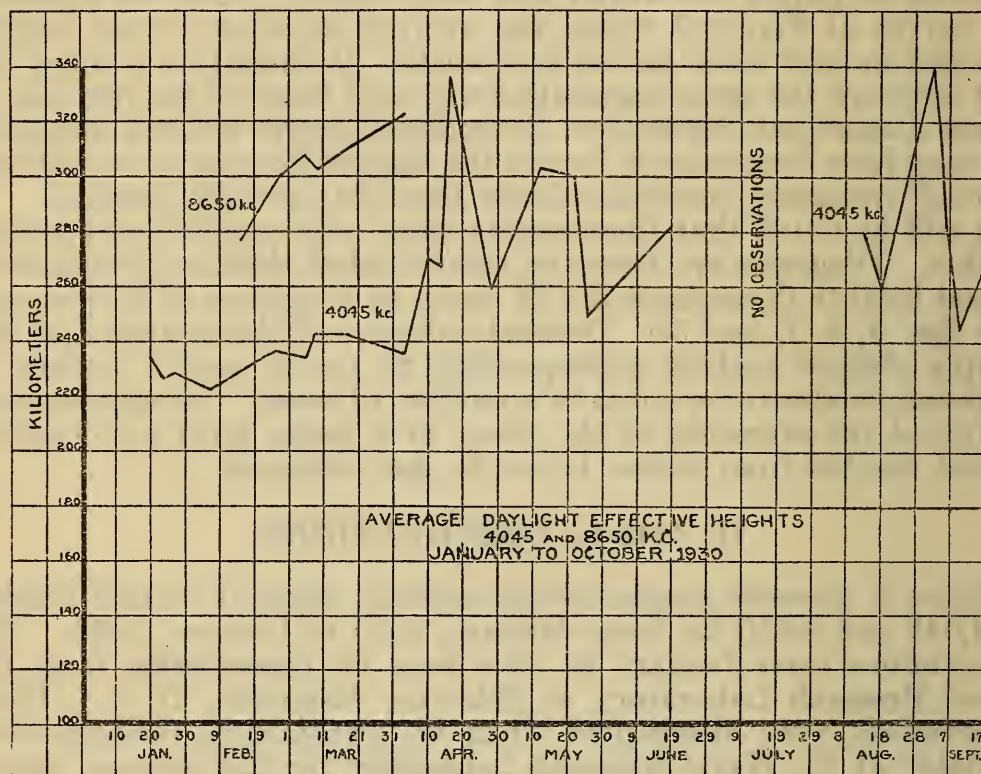


FIGURE 2.—*Average daylight (virtual) heights of the Kennelly-Heaviside layer from January to October, 1930 (4,045 and 8,650 kc)*

Note sky waves returned on 8,650 kc only between February and April of the period of observations.

100 km layer was observed during the daytime on 4,045 kc on August 15 and 16 (fig. 3) is, it will be noted, in marked contrast to the phenomena observable a few days later (August 20 and 21). (Fig. 4.) An interesting phenomenon in the form of a progressive downward movement of virtual height on 4,045 kc during the afternoon observed on August 25 and also on August 26, is also shown in Figure 4. This is one of the few cases where heights between 150 and 200 km were observable (that is, heights intermediate between the frequently occurring 100 to 130 and 250 to 300 km virtual heights referred to as the $E$ and $F$ regions or layers. [4] The pronounced low layer is clearly shown in the oscillogram of Figure 5, and the presence of a strong peak giving a high virtual height is shown in Figure 6.

The persistent appearance of large virtual heights not readily explicable on the basis of simple multiple reflections from the $E$ or $F$
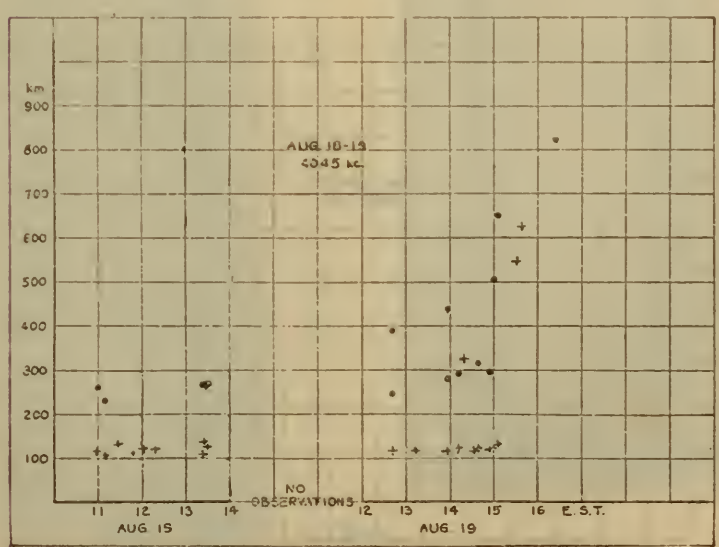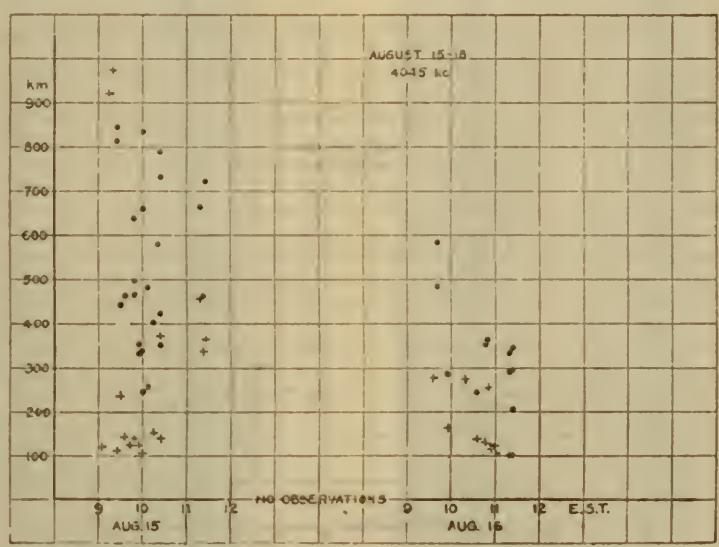
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[4] See footnote 4, p. 1084.
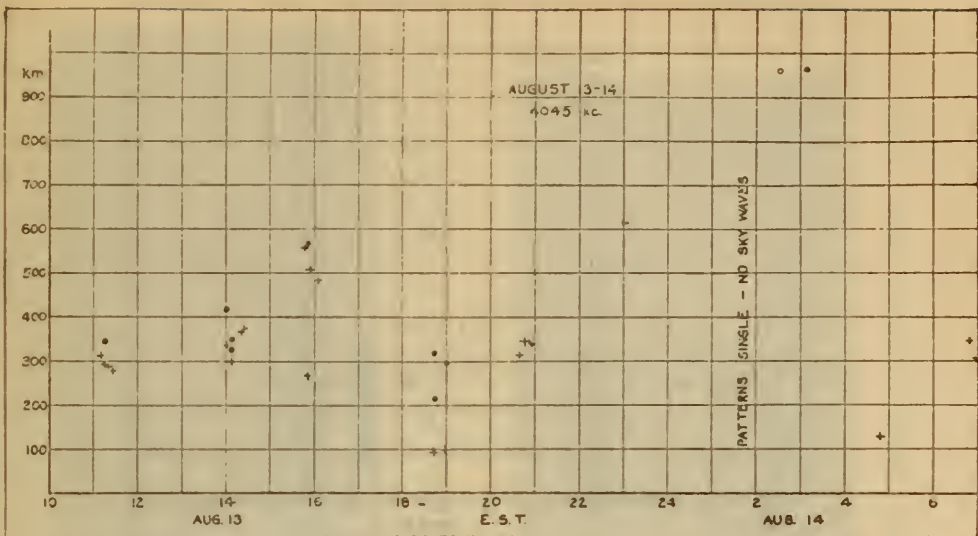
FIGURE 3.—*Scatter diagram of virtual heights observed during the period extending from August 13 to August 19, 1930*

Note frequent occurrence of 100 km heights.
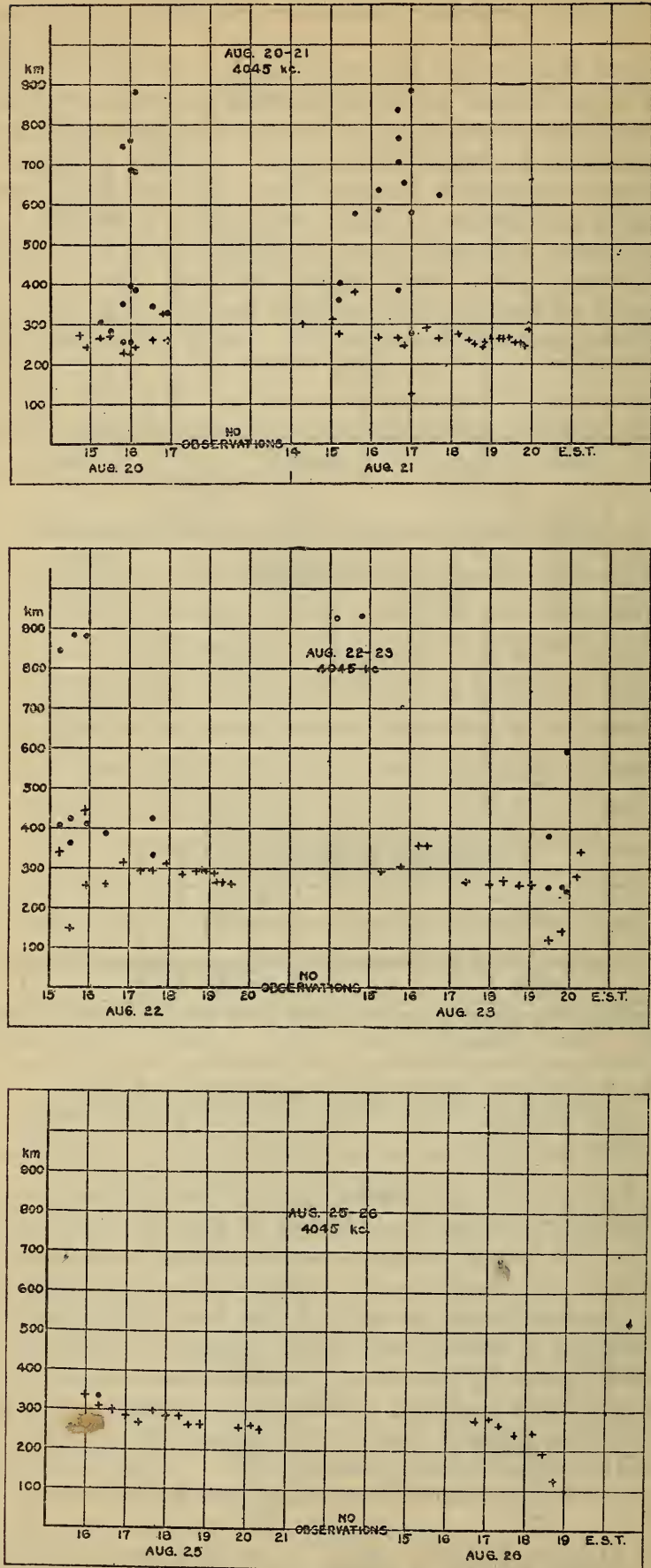
FIGURE 4.—*Scatter diagram of virtual heights observed during the period extending from August 20 to August 26, 1930* Note rare occurrence of 100 km heights.
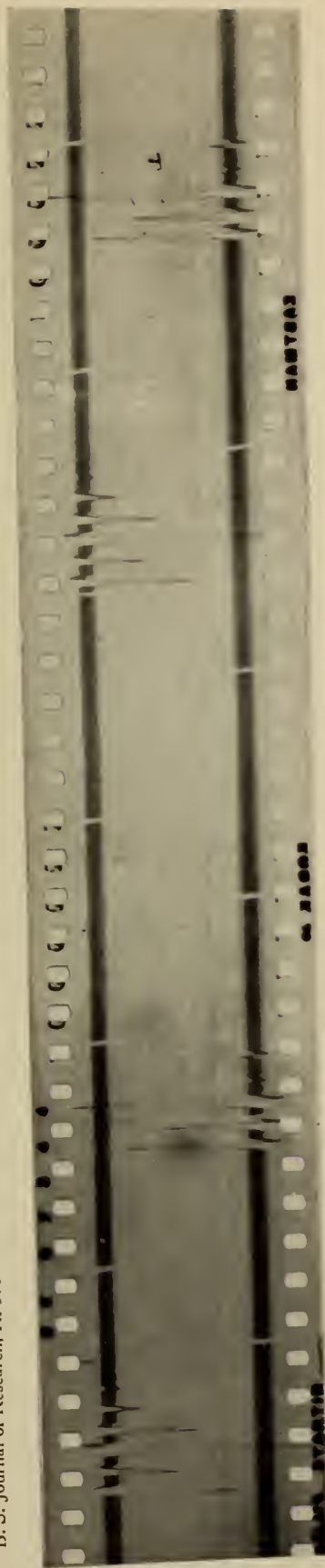
FIGURE 5.—*Upper and lower traces both show example of multiple reflections.* (*4,045 kc, 11.30 a. m., E. S. T., August 18, 1930*)



FIGURE 6.—*Note predominant 565 km virtual height. Note, however, faint evidence lower layer in upper trace.* (*4,045 kc, 3.50 p. m., E. S. T., August 13, 1930*)

layers, is emphasized in many of the oscillograms.  (See, for instance, figs. 1(*b*) and 6.)  Phenomena of this nature must doubtless be taken into account as the theory develops, for these phenomena are apparently quite typical.  A possible explanation is suggested by the theory presented in this paper.

## III. VIRTUAL LAYER HEIGHTS AS A FUNCTION OF FREQUENCY DURING DAYTIME

During August and September frequent observations were made on 1,600, 2,000, 3,000, 4,045, 5,000, 6,425, and 8,650 kc, approximately the same number of observations being taken on each frequency. The results obtained during daytime (9 to 15 e. s. t.) are shown in Figure 7.  No reflections were observed on 6,425 and 8,650 kc during
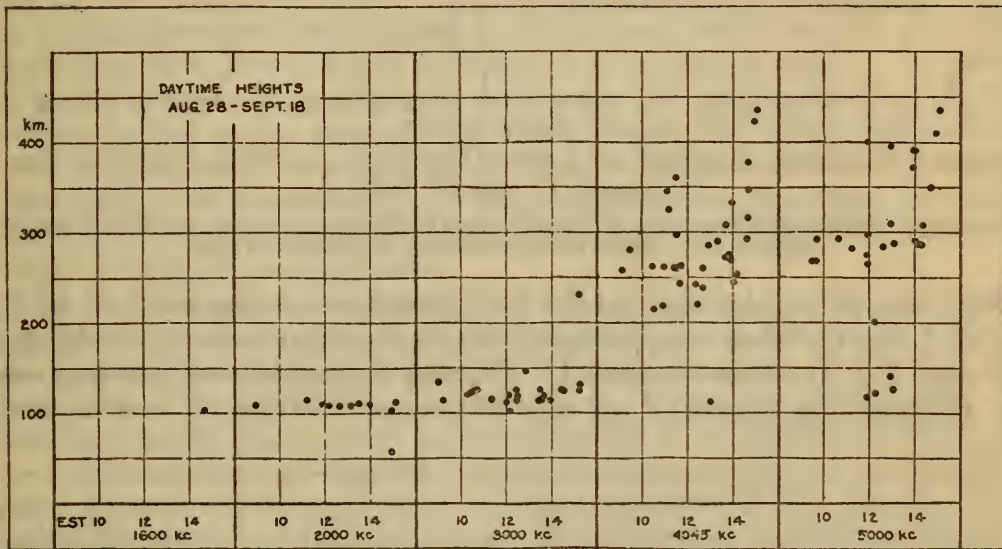


FIGURE 7.—*Daytime heights observed for 1,600, 2,000 3,000, 4,045, and 5,000 kc between August 28 and September 18, 930*

these hours.  This is in marked contrast to the conditions between February to April, when multiple reflections on 8,650 kc were frequently observed.

The patterns obtained for many of the observations, even on the lower frequencies, showed only the ground wave.  Figure 7 indicates the number of times that reflections occurred on the various frequencies.  It will be noted that virtual heights between 100 and 150 km, corresponding to the *E* layer, appear for 1,600, 2,000, and 3,000 kc. Occasionally this layer was observed on 4,045 and 5,000 kc, but virtual heights of over 200 km occurred most frequently on these higher frequencies.  Once on 3,000 kc and twice on 5,000 kc, coexisting *E* and *F* layers were observed.  Only the first, or in case of coexisting *E* and *F* layers, the first two reflections, were plotted in Figure 7.

## IV. DIURNAL VARIATION

Figure 8 shows the results obtained for frequencies of 1,600, 2,000, 3,000, 4,100, and 5,000 kc for a 24-hour period (September 17 to 18). The *E* layer was observed on 1,600 kc shortly after sunrise and just before sunset.  Otherwise no reflections were observed on this fre-

quency during daylight. However, just after sunset the virtual height was observed to be 232 km and during the night the height increased to 300 km. At sunrise it again dropped to the *E* layer region.

On 2,000 kc reflections were observable only three times during the 24 hours and the heights corresponded closely to those for 1,600 kc.
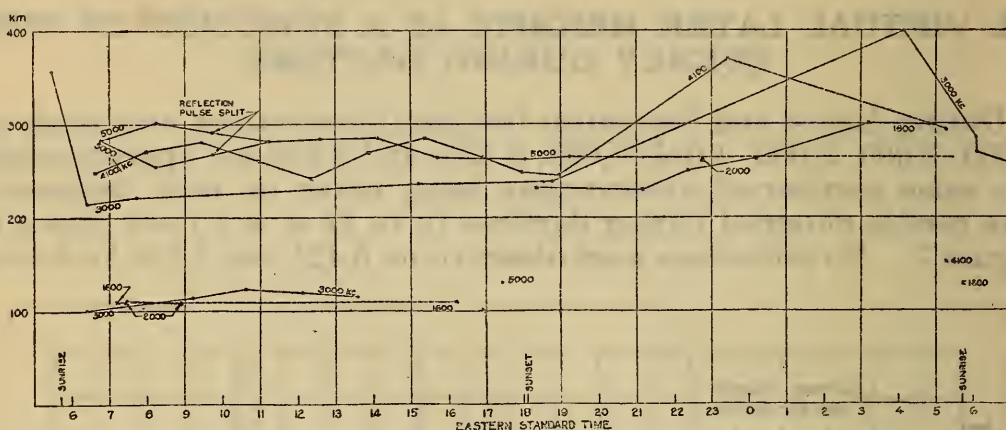


FIGURE 8.—*Diurnal changes on 1,600, 2,000, 3,000, 4,100, and 5,000 kc September 17 and 18, 1930*

The lowest distinct height from each oscillograph record is indicated by a cross, and distinct greater heights (not in simple multiple relation) are denoted by dots.

Just before sunrise the height for 3,000 kc was observed to be 357 km and shortly after sunrise coexisting reflections from the heights of 100 and 215 km were recorded. During the middle of the day only the *E* layer was observed on this frequency. After 15 e. s. t. the *E*



FIGURE 9.—*Diurnal changes on 1,600, 2,000, 3,000, 4,100, and 5,000 kc September 12, 1930*

The lowest distinct height from each oscillograph record is indicated by a cross, and distinct greater heights (not in simple multiple relation) are denoted by dots.

layer was no longer observable but reflections from the *F* region were again noted.

The behavior on 4,100 and 5,000 kc is clearly shown on the diagram. It will be noted that on 4,100 kc coexisting reflections from heights of 153 and 295 km were observed shortly before sunrise on the 18th.

Figure 9 shows diurnal variations for the same frequencies for a slightly shorter period on September 12.

# V. ANALYTICAL DISCUSSION OF RESULTS

## 1. REFRACTION

The phenomena observed in these tests offer evidence in favor of the existence of at least two more or less sharply defined strata in the upper atmosphere at which refraction takes place (as already suggested by other investigations conducted previously).[9,10] The refraction theory for one layer has already been discussed,[11,12] but it is of interest to consider the relations when two well-defined ionized strata exist. As Appleton has suggested, rays which are finally returned from the upper stratum nevertheless give virtual heights which may be considerably altered due to the transit through the lower layer. It is of interest to consider this possibility analytically to determine if such a view is in accord with observed phenomena. As in the case of a single layer, the theory requires an assumption as to the electron distribution in the layer, and the case of a single layer has been worked out for several such assumptions. At present it is, of course, impossible to represent the exact conditions which obtain in the upper atmosphere analytically, for these conditions are very imperfectly known and involve a turbulent rapidly varying distribution of ionization. It may be possible, however, to disclose some of the salient phenomena on the basis of simplified and necessarily idealized assumptions.

In the following discussion we will investigate such an idealized treatment to determine if it can be used to fit the facts. We will also investigate certain other possible explanations of some of these phenomena suggested as a result of those investigations. The simplest assumption, from the standpoint of analytical development, is a distribution of electron density, such as shown in Figure 10; that is, one which produces a linear change of the square of the index of refraction with height at the frequency under consideration. Under such conditions the path is (in the case of a single layer) parabolic. Several other assumed variations of electron density giving other types of trajectories have been investigated [13] and shown to yield rather similar results in the single layer case, provided the curves do not have a minimum index of refraction near zero. Under such conditions long retardations may be encountered.[14] These distributions, however, represent critical border line conditions and will not be considered here. In view of the above considerations, and with due regard to the simplicity attainable, the linear variation of squared index of refraction of the form shown in Figure 10 has been selected.

The postulated distribution considers a linear decrease in the square of the index of refraction starting at a height $E_0$ and decreasing linearly to a value $m^2$ at a height $E_1$.

The square of the index of refraction is then assumed to remain constant at the value $m^2$ until a height $F_0$ is reached, when it again starts to decrease linearly to a value zero at a height $F_1$.

[9] See footnote 4, p. 1084.
[10] See footnote 7, p. 1084.
[11] See footnote 5, p. 1084.
[12] P. O. Pedersen, The Propagation of Radio Waves, published by Denmarks Naturvioenkabeliage Samfund. Copenhagen; 1928.
[13] See footnote 5, p. 1084.
[14] G. Breit, Group Velocity and Retardation of Echoes, Proc. I. R. E., vol. 17, pp. 1508–1512; September, 1929.

The layers are suggested schematically in Figure 11 where the equations corresponding to the assumed linear deviations are also indicated. A curve showing the trajectory and indicating the nota-
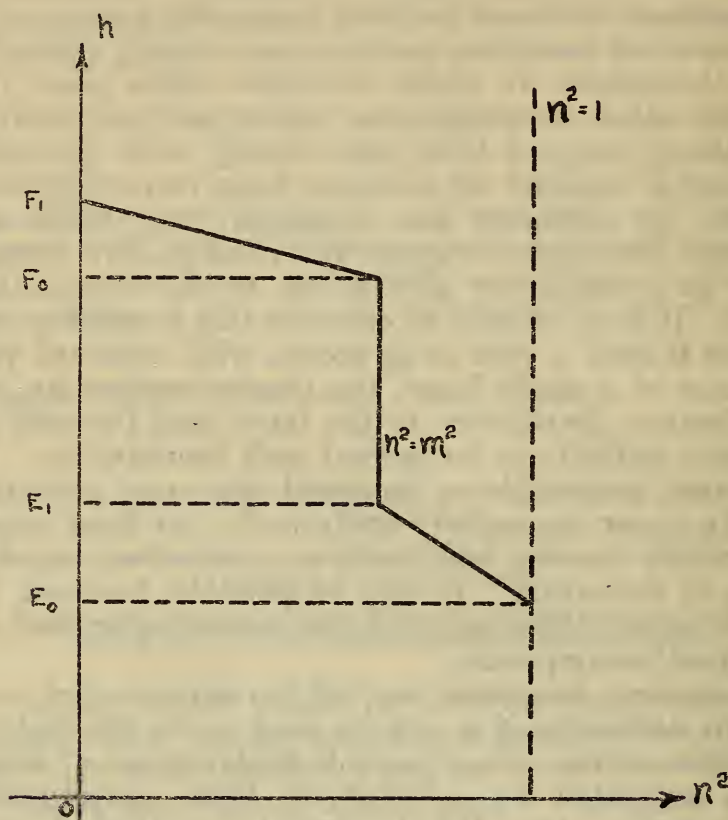


FIGURE 10.—*Assumed change of index of refraction squared as a function of height*

tion adopted is given in Figure 12. In carrying out the computations of virtual height for this case, free use will be made of the notation and the results evolved in a previous paper.[15]
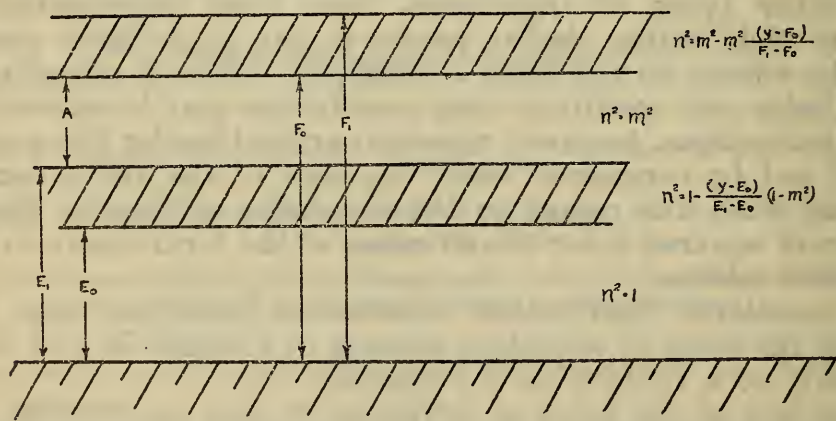


FIGURE 11.—*Assumed layer distribution showing equations for variation of n*

In computing the virtual height $h'$ we will avail ourselves of Breit and Tuve's theorem (1) which tells us that in the absence of dissipation (for any assumed variation of index of refraction with height),

[15] See equation (V–1), p. 720, Kenrick and Jen, Measurement of the Height of the Kennelly-Heaviside Layer, Proc. I. R. E., vol 17, pp. 711–733; April, 1929.

the group retardation is the same as that encountered by a ray traveling with the velocity of light, and traversing a triangular path with the given base and given angle of departure. Hence (see fig. 12 for notation),

$$X_0 = 2 \, h' \tan \phi_0 \qquad (1)$$

This gives us a relation between the virtual height $h'$, the angle of departure $\phi_0$, and the base $X_0$.

We will now compute the relation between the coordinate $\dfrac{X_0}{2}$ and

$\phi_0$ by adding together the $x$ intervals corresponding to each part of the path. This will enable us to compute the relation between actual height of the strata, the virtual height, $m$, $\phi_0$, etc.
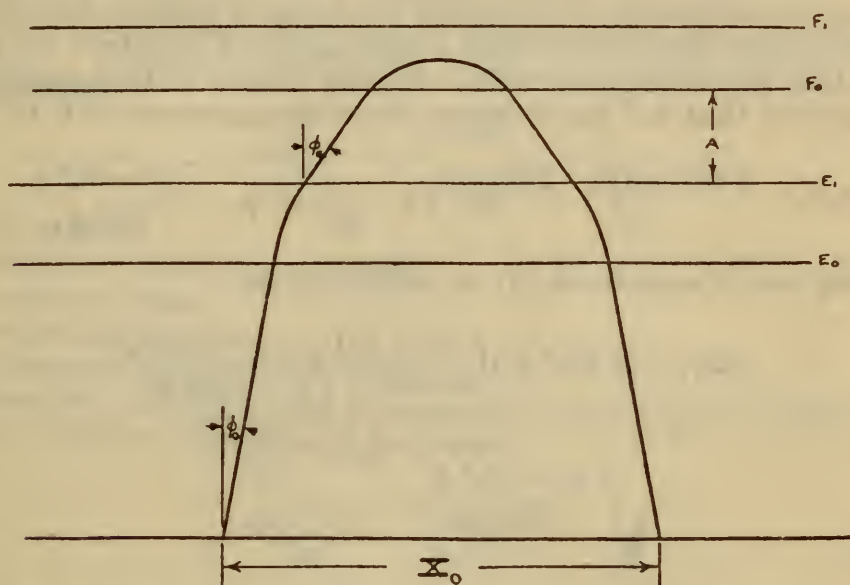


FIGURE 12.—*Form of trajectory of ray in presence of two layers*

The ray moves in a straight line as far as the height $E_0$; hence its horizontal displacement thus far is merely

$$(\Delta x) \,_{E0} = E_0 \tan \phi_0 \qquad (2)$$

The ray travels in a parabolic arc through the $E$ layer. Its horizontal displacement during this part of its motion is [16]

$$(\Delta x) \,_{E_1 - E_0} = \frac{2 \sin^2 \phi_0}{1 - m^2} (E_1 - E_0) \left\{ \cot \phi_0 - \sqrt{\cot^2 \phi_0 - \frac{(1 - m^2)}{\sin^2 \phi_0}} \right\} \qquad (3a)$$

or

$$(\Delta x) \,_{E_1 - E_0} = \frac{2 \sin \phi_0 \cos \phi_0}{1 - m^2} (E_1 - E_0) \left( 1 - \sqrt{1 - \frac{1 - m^2}{\cos^2 \phi_0}} \right) \qquad (3b)$$

The ray leaves the $E$ layer at angle given by the Snell law and the differential equation of the ray path (see reference 11; Equation III–1); that is

$$\sin \phi_E = \frac{\sin \phi_0}{m} \qquad (4a)$$

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[16] See footnote 14, p. 1091.

from equation (4a) we have, by elementary trigonometry, that

$$\cos \phi_E = \frac{\sqrt{m^2 - \sin^2 \phi_0}}{m} \qquad (4b)$$

and hence

$$\tan \phi_E = \frac{\sin \phi_0}{\sqrt{m^2 - \sin^2 \phi_0}} \qquad (4c)$$

Using equations (4a) and (4c), and denoting the distance between the layers $(F_0 - E_1)$ by $A$, we have immediately that the $x$ displacement of the ray in passing from $E_1$ to $F_0$ is:

$$(\Delta x)_{F_0 - E_1} = (F_0 - E_1) \tan \phi_E = \frac{(A) \sin \phi_0}{\sqrt{m^2 - \sin^2 \phi_0}} \qquad (5)$$

Noting from equation (4) that the thickness of penetration $(Y_F - F_0)$ into the $F$ layer is until $n^2 = \sin^2 \phi_E$, application of equation ($V$–1a, reference 11) to this case gives us (in a manner similar to that employed in deducing (3)) that for the $F$ layer, the $x$ displacement will be

$$(\Delta x)_F = \frac{2 \sin^2 \phi_E (F_1 - F_0)}{m^2} \left\{ \cot \phi_E - \sqrt{\cot^2 \phi_E - \frac{m^2 - \sin^2 \phi_E}{\sin^2 \phi_E}} \right\} \qquad (6a)$$

or making use of equations (4) to eiliminate $\phi_E$

$$(\Delta x)_F = (2 \sin^2 \phi_0) \left( \frac{(F_1 - F_0)}{m^2} \right) \left\{ \frac{\sqrt{m^2 - \sin^2 \phi_0}}{\sin \phi_0} \right.$$
$$\left. - \sqrt{\frac{m^2 - \sin^2 \phi_0}{\sin^2 \phi_0} - \frac{m^2 - \frac{\sin^2 \phi_0}{m^2}}{\frac{\sin^2 \phi_0}{m^2}}} \right\} \qquad (6b)$$

or

$$(\Delta x)_F = \frac{2 \sin \phi_0 (F_1 - F_0) (\sqrt{m^2 - \sin^2 \phi_0})}{m^4} [1 - \sqrt{1 - m^2}] \qquad (6c)$$

Adding together equations (2), (3b), (5), and (6) to obtain the total $\frac{x_0}{2}$ (and utilizing equation (1) involving the virtual height,) we have:

$$\frac{x_0}{2} = h' \tan \phi_0 = E_0 \tan \phi_0 + (2 \sin \phi_0 \cos \phi_0) \frac{E_1 - E_0}{1 - m^2} \left( 1 - \sqrt{1 - \frac{1 - m^2}{\cos^2 \phi_0}} \right)$$
$$+ A \frac{\sin \phi_0}{\sqrt{m^2 - \sin^2 \phi_0}} + \frac{2 \sin \phi_0 (F_1 - F_0) \sqrt{m^2 - \sin^2 \phi_0}}{m^4} [1 - \sqrt{1 - m^2}] \qquad (7a)$$

Hence

$$h' = E_0 + 2 \cos^2 \phi_0 \left( 1 - \sqrt{1 - \frac{1 - m^2}{\cos^2 \phi_0}} \right) \frac{E_1 - E_0}{1 - m^2} + \frac{A \cos \phi_0}{\sqrt{m^2 - \sin^2 \phi_0}}$$
$$+ \frac{2 (F_1 - F_0) \cos^2 \phi_0 \sqrt{m^2 - \sin^2 \phi_0} [1 - \sqrt{1 - m^2}]}{m^4} \qquad (8)$$

So far, no approximations have been made, and equation (8) may be used to solve for $h'$ for any $x$ (by successive approximations).

However, when $X_0$ is small; that is, when we are working at nearly normal incidence, and when $m^2$ is small with respect to unity but large with respect to $\sin^2\phi_0$, equation (8) may be written approximately as

$$h' = E_0 + \frac{2(E_1 - E_0)}{1 + m} + \frac{A}{m} + \frac{F_1 - F_0}{m} \tag{9a}$$

If now we define the critical frequency as that frequency for which the index of refraction of the $E$ layer is zero, then for slightly higher frequencies $m$ will be small and the effective distance between layers may be increased to several times its actual value as may be seen from equation (9a). This effect, then, offers a possible explanation for the erratic heights observed for the $F$ layer.

The relation between $m$ and frequency may be written in a convenient form for nondissipative media (which, of course, are assumed throughout). Thus, the refractive index $n$ is given by the well-known equation: [17]

$$n^2 = \epsilon = 1 - \frac{Ne^2}{m_1 \pi f^2} \tag{10}$$

where

$\quad f =$ frequency,
$\quad \epsilon =$ effective dielectric constant,
$\quad m_1 =$ electron mass,
$\quad e =$ electron charge,
$\quad N =$ electron density in electrons/cc.

Thus from our definition and by equation (10) we get the following expression involving the critical frequency:

$$n^2 \equiv 0 = 1 - \frac{N_{(max)E}e^2}{m_1 \pi f_c^2} \tag{11a}$$

or

$$f_c^2 = \frac{N_{(max)E}e^2}{m_1 \pi} \tag{11b}$$

where $N_{(max)E}$ is the maximum electron density of the $E$ layer. $f_c$ is the highest frequency for which energy is returned by refraction at normal incidence from a nonturbulent layer of the type assumed in this analytical investigation.

On substituting $f_c^2$ for $\dfrac{N_{(max)E}e^2}{m_1 \pi}$ in equation (10) the square of the index of refraction at the highest point of the $E$ layer is given by:

$$m^2 \equiv n^2_{(min)E} = 1 - \left(\frac{f_c}{f}\right)^2 \tag{11c}$$

If for a particular time we take $f_c = 3,750$ kc, then at 4,000 kc, $m = \sqrt{1 - \left(\dfrac{3.75}{4}\right)^2} = 0.35$ and the distance between layers indicated by height measurements would be about three times too great. Thus this critical value indicates an explanation for the virtual heights of 500 km or more, which were frequently observed during the tests.

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[7] See footnote 12, p. 1091.

The critical frequency, which is proportional to the square root of the electron density, is extremely variable, and doubtless changes considerably from hour to hour.

The theory developed above would permit the coexistence of $E$ and $F$ layer reflections only very near the critical frequency. Thus just below the critical frequency no energy will get through the $E$ layer for a given angle of incidence and a reflection from this layer will occur. However, at a slightly smaller angle of incidence energy can get through the $E$ layer and be reflected from the $F$ layer. During the tests described such coexisting reflections were observed at various times on 3,000, 4,100 and 5,000 kc, so that the value of $f_c$ chosen may be considered as a fair example for calculating the maxi-
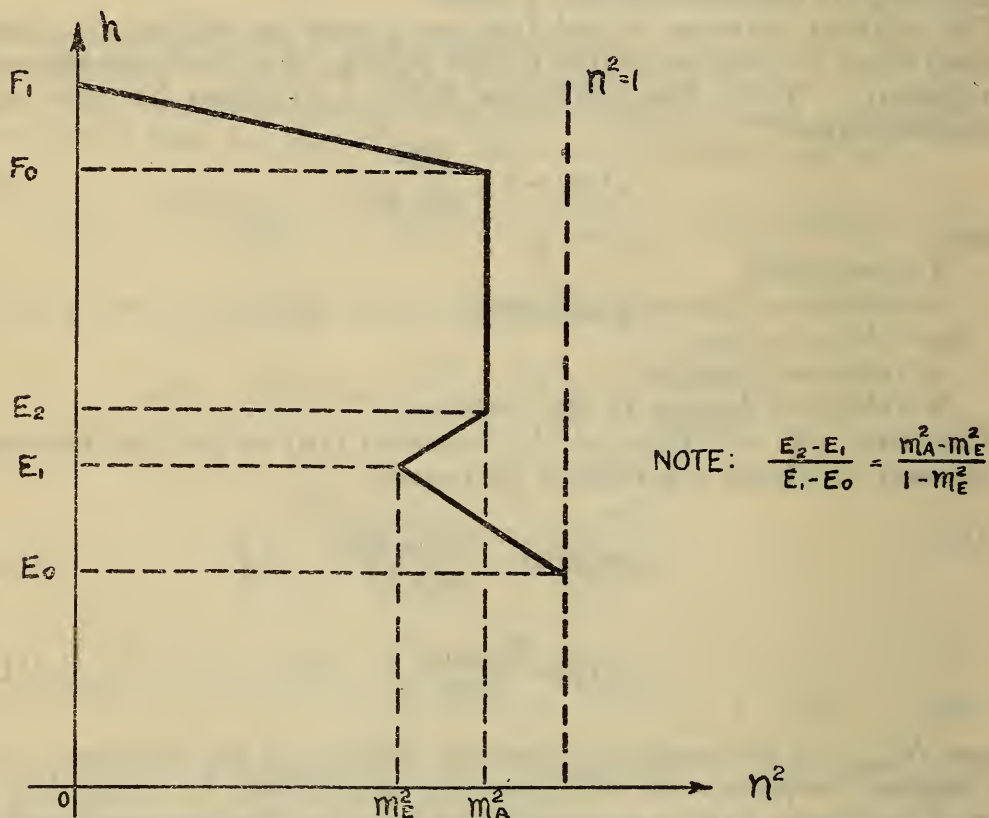


NOTE: $\dfrac{E_2 - E_1}{E_1 - E_0} = \dfrac{m_A^2 - m_E^2}{1 - m_E^2}$

FIGURE 13.—*More general form of assumed in variation showing minimum value in first layer*

mum electron density in the $E$ layer by means of equation (11b). This calculation gives

$$N_{MAX} \ (E \ \text{layer}) = 1.7 \times 10^5 \ \text{electrons per cc.}$$

A similar computation for the summer $F$ layer, assuming value $f_c = 5,500$ kc, gives

$$N_{MAX} \ (F \ \text{layer}) = 3.8 \times 10^5 \ \text{electrons per cc.}$$

Little specific information is available as to the electron distribution in the upper atmosphere, but it is perhaps appropriate to consider a slight generalization of the above analysis, for, thus far, it will be noted we have considered the reduction of $n^2$ with height to be progressive; that is, no maximum and subsequent minimum of

$N$ was considered between the layers. A more general type of variation to which the above analysis may be readily extended is obtained by an assumption such as indicated in Figure 13.

This variation differs from that of Figure 10 only in that the $n^2$ variation is assumed to reach a minimum $m^2_E$ in the $E$ layer and then $n^2$ is again assumed to increase (along a symmetrical straight line of equal slope) to a value $m^2_A$, where it remains until the height $F_0$ is attained. The equation of the variations are indicated in Figure 14, and the generalization of equation (7) to include this case gives

$$\frac{x_0}{2} = h' \tan \phi_0 = E_0 \tan \phi_0 + 2\left[ 2 \sin \phi_0 \cos \phi_0 \frac{E_1 - E_0}{1 - m^2_E}\left(1 - \sqrt{1 - \frac{1 - m^2_E}{\cos^2\phi_0}}\right)\right]$$

$$-2 \sin \phi_0 \cos \phi_0 \frac{(E_1 - E_0)}{1 - m^2_E}\left(1 - \sqrt{1 - \frac{1 - m^2_A}{\cos^2\phi_0}}\right) + \frac{[A - (E_2 - E_1)] \sin\phi_0}{\sqrt{m^2_A - \sin^2\phi_0}}$$

$$+ \frac{2 \sin \phi_0 (F - F_0) \sqrt{m_A{}^2 - \sin^2\phi_0}}{m_A{}^4} [1 - \sqrt{1 - m_A{}^2}]$$
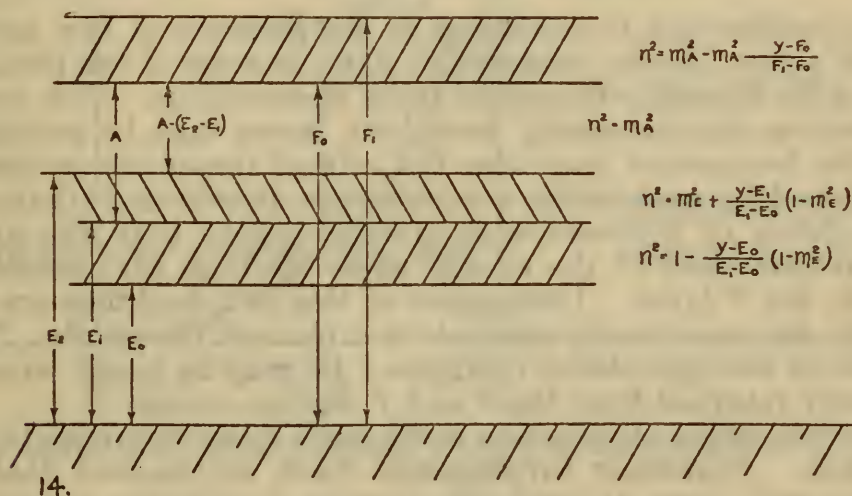


FIGURE 14.—*Layer distribution corresponding to more general variation of n indicated in Figure 13*

In writing this equation we note that, for this case, by symmetry, the parabolic path $(\Delta x)E_0 - E_2$ would be just twice its value in equation (7a) provided $m_A{}^2 = 1$. For general variations corresponding to other values of $m^2_A$, $(\Delta x)E_0 - E_2$ is twice that of equation (7a) minus the $(\Delta x)$ corresponding to the change from $n^2 = 1$ to $n^2 = m_A{}^2$ along the parabolic path; this is represented by the subtractive term of equation (7b).

The approximations previously employed in deducing equation (9) from equation (7) may be applied (subject to the same approximations) to deduce from equation (13) the value of virtual heights $h'$ for the generalized case, that is,

$$h' = E_0 + \frac{2(E_1 - E_0)}{1 - m_E{}^2}(1 - 2m_E + m_A) + \frac{A - (E_2 - E_1)}{m_A} + \frac{F_1 - F_0}{m_A} \quad (9b)$$

It will be noted that when $m_A = m_E$ and $(E_2 - E_1) = 0$, equations (7b) and (9b) reduce, respectively, to (7a) and (9a).

The nature of the phenomena indicated by equations (7b) and (9b) is not essentially different from that previously discussed, but indicates that for a given $F_1 - E_0$ and $m_E$ the variation in virtual height of the $F$ layer produced by the $E$ layer is greatest when the maximum $m_A^2$ between the layers is small (that is, when there is no pronounced decrease in $N$ between the layers). The values indicated in the discussion of equation (9) are thus upper bounds (at least for the types of variations assumed). Previous investigations have indicated the conclusions to be independent of the exact form of variations assumed,[18] except for certain very special conditions,[19] and the relations deduced above may hence be considered as reasonably representative. In general, we would expect the value of $m_A$ to be distinctly less than unity, so the phenomena discussed will be in evidence, in general, near the critical frequency for the $E$ layer. (Note that if $m_A$ is not considerably less than unity, equation (7b) should be used, as equation (9b) is no longer a good approximation).

## 2. REFLECTION PHENOMENA

The consideration of refraction in the presence of two layers, as outlined above, throws considerable light on some of the phenomena observed in Kennelly-Heaviside layer observations. The presence of reflections simultaneously from both layers may be explained as above for frequencies very near the critical frequency or they may be explained by postulating a nonuniform distribution (that is, turbulent clouds of electrons with greater density than the average) which return some of the energy while allowing the remainder to travel to the $F$ layer. Phenomena of this sort doubtless are of importance, but seem hardly adequate to explain all the results. Numerous cases of the type shown in Figure 1 ($b$) may be found, where rays apparently returned from the $E$ and $F$ regions, coexist.

An investigation of reflection coefficients seems appropriate in this connection. Numerous investigators have emphasized that little reflection will occur at gradually varying boundaries provided the first and second derivatives of the variation are not large and do not exhibit discontinuities, in short, provided the variation extends over a number of wave lengths and is free from abrupt stratifications.[20] [21] [22]

It must be remembered, however, that the power returned, is usually small and so variable that reflection coefficients of a few per cent would be sufficient to explain many of the observed phenomena. It therefore seemed appropriate to review quantitatively the reflection theory with a view to establishing how much this phenomenon might be invoked to explain the results. In this paper we will confine ourselves for simplicity to the case corresponding to most of the observations; that is, that of normal incidence.

Lord Rayleigh's work is notable among the investigations of this problem of the reflection at normal incidence from gradually varying media. He obtained solutions directly from the differential equation of the motion (for certain special assumed laws of variations of the medium which rendered the resulting equation integrable).[21] He

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[18] See footnote 5, p. 1084.
[19] See footnote 14, p. 1091.
[20] See footnote 12, p. 1091.
[21] Rayleigh (Strutt, J. W.), On Reflection of Vibrations at the Confines of Two Media Between Which the Transition is Gradual, Proc. London Math. Soc., pp. 51-56; 1881.
[22] Rayleigh (Strutt, J. W.), On the Propagation of Waves Through a Stratified Medium with Special Reference to the Question of Reflection. Collected Works, vol 6, pp. 71-90.

also obtained the solution as a limiting case of the solution for stratified transitions as the number of stratifications became infinite and the amplitude of the transitions between any two strata infinitesimal.[23]

The most general (and for our purposes, the most convenient) relation he derives is to be found in the second paper. (See ref. 22, p. 80, equation (5).) Rayleigh's investigations are not limited to waves in optical media, but, as he points out, may be readily reduced
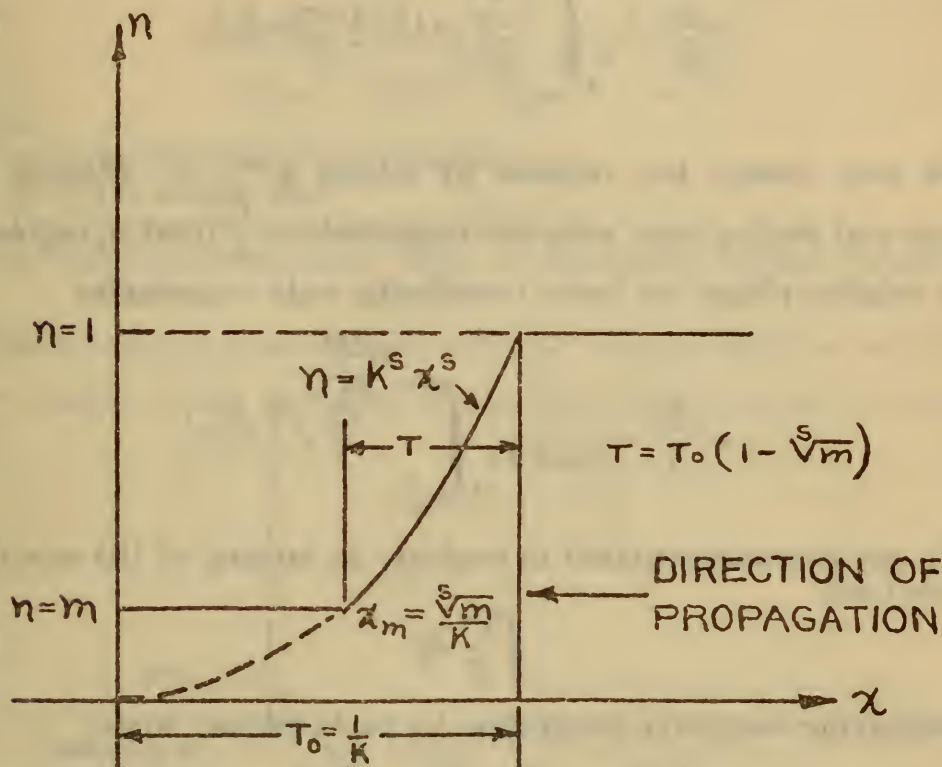


FIGURE 15.—*Form of transition assumed in investigation of reflection.*
(*Case I*)

to this case. When so reduced, this equation may be written in our notation as:

$$\frac{B_1}{A_1} = - \int \frac{dn}{2n} \, \varepsilon^{-2j \int_0^x \frac{2\pi n}{\lambda n_0} dx} \tag{12}$$

where

    $n$  = index of refraction at any point $x$ in the variable medium.
    $\lambda$  = wave length in vacuo.
    $j$  = $\sqrt{-1}$
    $B_1$ = amplitude of reflected wave.
    $A_1$ = amplitude of incident wave.
    $r$  = reflection coefficient at normal incidence.

It should be noted that $\left|\dfrac{B_1}{A_1}\right| = r$. This relation will be used in this investigation.

The results, of course, depend on the assumed type of variation of dielectric constant, but an idea of the magnitude to be expected may be obtained by investigating some plausible distributions.

We will now proceed to evaluate equation (12) for rather general types of variation of $n$ with $x$.

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[23] See footnote 22 p. 1098.

*Case I.*—We will first consider a variation of the form $n = k^s x^s$. (See fig. 15.)

Substituting this value of $n$ in equation (12) we have for the reflection (for a strata varying from an $n = 1$ to an $n = m$ along the assumed curve)

$$\frac{B_1}{A_1} = -\int_{x=\frac{1}{k}}^{x=\sqrt[s]{\frac{m}{k}}} \frac{s\,dx}{2x}\, \epsilon^{j\left(\frac{4\pi}{\lambda}\right)\left[\frac{k^s x^{s+1}}{s+1} - \frac{k^s}{s+1}\right]} \tag{13}$$

We may change the variable by letting $x^{s+1} = z$. Making this change and dealing only with the magnitude of $\frac{B_1}{A_1}$ (that is, neglecting their relative phase) we have, considering only magnitudes

$$r = \left|\frac{B_1}{A_1}\right| = \left|\frac{s}{2(s+1)}\int_{z=\frac{1}{k^{s+1}}}^{z=\frac{m^{\frac{s+1}{s}}}{k^{s+1}}} \frac{dz}{z}\, \epsilon^{j\frac{4\pi}{\lambda}\frac{k^s z}{(s+1)}}\right| \tag{14}$$

We are thus only required to evaluate an integral of the relatively simple form

$$\int \frac{\epsilon^{ay}}{y}\,dy \tag{15a}$$

Employing successive integration by parts we may write:

$$\int \frac{\epsilon^{ay}}{y}\,dy = \frac{\epsilon^{ay}}{ay} + \frac{\epsilon^{ay}}{a^2 y^2} + 2\int \frac{\epsilon^{ay}}{a^2 y^3}\,dy \tag{15b}$$

For real values of $y$ and pure imaginary values of $a$ this series may be written in the asymptotic form:

$$\left|\int \frac{\epsilon^{ay}}{y}\,dy\right| = \left|\frac{\epsilon^{ay}}{ay} + \frac{\epsilon^{ay}}{a^2 y^2}\right| + 2R\int \frac{dy}{a^2 y^3} \qquad -1 \leq R \leq 1 \tag{15c}$$

Remembering that $a$ is a pure imaginary, we note that the integral in general represents a rapidly pulsating function but that always

$$\left|\int \frac{\epsilon^{ay}}{y}\,dy\right| \leq \left|\frac{1}{ay}\right| + \left|\frac{1}{a^2 y^2}\right| + \left|\frac{1}{a^2 y^2}\right| \tag{15d}$$

Applying (15d) to integral 14, and further approximating by neglecting phase differences at the limits, we may write further the order of magnitude of $\left(\frac{B_1}{A_1}\right)$ by the inequality

$$r = \left|\frac{B_1}{A_1}\right| \leq \frac{s}{2(s+1)}\left\{\left|\frac{(s+1)k\lambda}{4\pi}\left(\frac{1}{m^{\frac{s+1}{s}}} + 1\right)\right.\right.$$

$$\left.\left. + 2\frac{(s+1)^2 k^2 \lambda^2}{16\pi^2}\left(\frac{1}{m^{\frac{2(s+1)}{s}}} + 1\right)\right|\right\} \tag{16}$$

For $\dfrac{1}{m}$ large compared with unity and $\dfrac{(s+1)\lambda k}{4\pi m^{\frac{s+1}{s}}}$ small compared with unity, this inequality approaches an equality to terms of the order of $r^2$, the function ceases to oscillate violently, and we may write (noting further that $\dfrac{T}{T_0}=\left(1-m^{\frac{1}{s}}\right)$)

$$r \doteq \frac{sk\lambda}{8\pi m^{\frac{s+1}{s}}} = \frac{s\lambda}{8\pi m^{\frac{s+1}{s}} T_0} = \frac{s\lambda}{8\pi T} \frac{\left(1-m^{\frac{1}{s}}\right)}{m^{\frac{s+1}{s}}} \tag{17}$$

where

    $s$ = exponent of law of variation.
    $\lambda$ = wave length (*km*).
    $T$ = thickness of varying layer (*km*).
    $m$ = minimum refractive index $n$ encounted during variation in layer.

This coefficient does decrease with $\dfrac{\lambda}{T_o}$; that is, with the ratio of the wave length to the thickness of the layer (extrapolated to zero $n$),
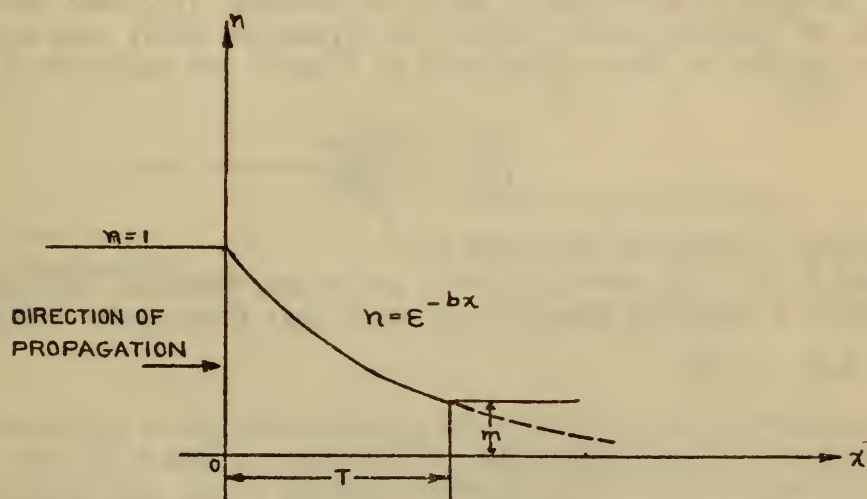


FIGURE 16.—*Form of transition assumed in investigation of reflection* (*Case II*)

but it will be noted the decrease is only linear; that is, for small coefficients of reflection the coefficient $r$ is inversely proportional to the first power of the thickness of the layer. This coefficient varies (but not critically) with the powers of the variation of $n$ with $x$. Thus, for a linear variation ($s=1$) we have, taking $T=10$ *km*, $\lambda=0.1$ *km*, and $m=0.3$; $\dfrac{s(1-m)}{8\pi(m^2)}=0.3$ and $r=0.003$. For $s=2$, the other data remaining the same, we have $\dfrac{s(1-m^{\frac{1}{2}})}{8\pi(m^{\frac{1}{2}})}=0.2$ and $r=0.002$; likewise for $s=3$, $r=0.002$.

While these coefficients of reflection are small, they are nevertheless quite appreciable, and reflections thus produced should be quite observable. It will be noticed further that close to the critical frequency much larger values of $r$ may be found due to the small values of $n$ encountered in that region. (However, note that the equation for $r$ is, of course, not valid unless $r^2 < < r$.)

*Case II.*—$n = \epsilon^{-bx}$. We will next consider an assumed variation of the form $n = \epsilon^{-bx}$.

As will be noted by reference to Figure 16, we will consider an exponential variation starting at $x = 0$ and extending to $x = T$, where the minimum index of refraction ($n = m$) is encountered.

This gives $\epsilon^{-bT} = m$ or $b = \dfrac{\log_E \dfrac{1}{m}}{T}$

Equation (12) when applied to this case gives:

$$r = \left| \frac{B_1}{A_1} \right| = \left| \frac{b}{2} \int_{x=0}^{x=T} \epsilon^{j \frac{4\pi \epsilon^{-bx}}{\lambda b}} \, dx \right| \tag{18}$$

or, letting $\epsilon^{-bx} = y$

$$r = \left| \frac{1}{2} \int_{y=1}^{y=m} \frac{dy}{y} \epsilon^{j \frac{4\pi y}{\lambda b}} \right| \tag{19}$$

This integral is identical in form to equation (14) and the same methods of solution hold; hence, by equation (15b) and approximations similar to those employed in Case I we have for $r$ small and $m \ll 1$.

$$r = \left| \frac{B_1}{A_1} \right| = \frac{\lambda \log_\epsilon m}{8\pi T m} \tag{20}$$

which holds to terms of the order of $r^2$.

Taking $\lambda = 0.1$ *km*, $m = 0.3$, $T = 10$ as before, equation (20) gives $r = 0.001$, a result in good accord with that previously obtained.

*Case III.*—$n = \dfrac{P}{X}$.

It is of interest, in conclusion, to compare the above solutions with that obtained by Rayleigh in his earlier paper. (See p. 55, ref. (21).) This solution is of particular interest because it was not obtained by use of equation (12), but by a direct investigation of the wave equation, which is solvable in this special case. (It will be noted that $s = 1$ is excluded from our $s$ power solution and has to be handled separately, due to the logarithm obtained on integrating $\dfrac{dx}{x}$.) Comparison with our results obtained above requires only a reduction of his general solution for any motion satisfying the wave equation to our special notation.

Rayleigh's problem, when reduced to the optical case and our notation, is indicated in Figure 17. Propagation is assumed in a positive direction and the variability to extend between $x = x_1$ and $x = x_2$ (we have taken $x_1 = 1$ and hence $x_2 = \dfrac{1}{m}$ which, if $T$ is the desired layer thickness in *km*, necessitates a unit of length of $\dfrac{T}{\dfrac{1}{m} - 1}$ *km*).

In these units $\lambda$ is related to $T$ in $km$ by $\lambda = (\lambda \, km)\left(\dfrac{\frac{1}{m}-1}{T}\right)$. In ex-
amining units in the solution as given by Rayleigh, care must be
taken not to confuse our $m$ (minimum value of $n$) with the $m$ he uses,
which we will designate by $m'$ to avoid confusion.

$$
m' = \sqrt{\left[\frac{2\pi T}{\lambda_{km}\left(\frac{1}{m}-1\right)}\right]^2 - \frac{1}{4}}
$$

In our notation, Rayleigh's solution is

$$
r = \left|\frac{B_1}{A_1}\right| = \sqrt{\frac{\sin^2\left(m'\log\frac{1}{m}\right)}{4m'^2 + \sin^2\left(m'\log\frac{1}{m}\right)}} \tag{21}
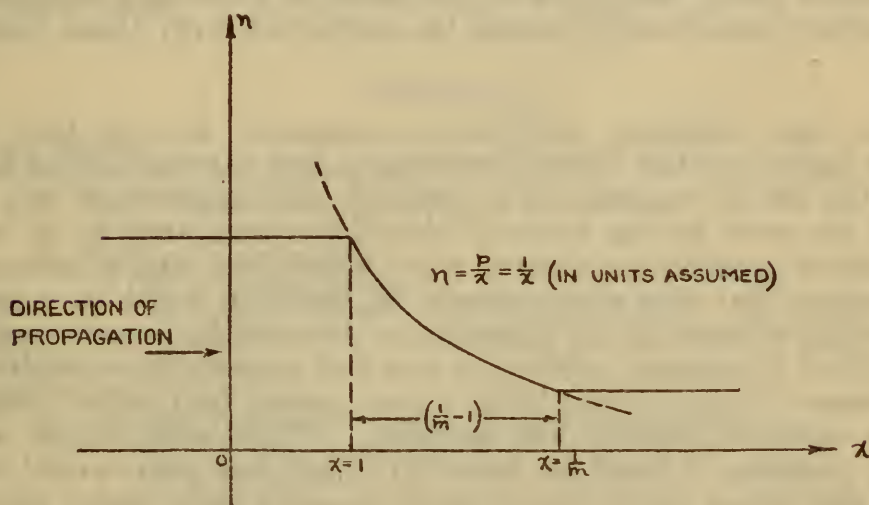$$



FIGURE 17.—*Form of transition assumed in investigation of reflec-
tion.* (*Case III*)

It is interesting to note that this solution pulsates rapidly from
zero to maxima of the order of:

$$(m \angle \angle 1)\ (r \angle \angle 1)$$

$$
r \doteqdot \frac{\lambda_{km}}{4\pi T_{km}m} \tag{22}
$$

This upper bound is a result of the same order of magnitude as
obtained in Cases I and II. An application of the integral given in
equation (12) will indicate this property, for in this case changes of
variable reduce it to a form involving an integral between limits of
the form

$$
K\left|\int_A^x \varepsilon^{jk\theta}d\theta\right|
$$

which does not involve an inverse power of $\theta$ multiplying the exponen-
tial, and hence continues to pulsate reading zero at frequent intervals

over all finite but large ranges of $x$.   Space will not be taken here for a complete discussion of this case, as the result is obtained by Rayleigh in an independent method which offers a further check on the order of magnitude of our results obtained in the other cases.   Thus equation (22) gives for $\lambda = 0.1 \ km$, $T = 10 \ km$, and $m = 0.3$ on upper bound of $r$ of

$$r = 0.003$$

## VI. CONCLUSIONS

The results of this paper indicate that the phenomena observed may be profitably investigated both from the point of view of refraction in the presence of two layers and from the point of view of reflection (particularly near the critical frequency $f_c$ where the minimum value of the effective index of refraction nearly attains zero).   Theoretical predictions that the most marked effects due to these causes are to be noted near the critical regions, are apparently justified by the experiments reported, which also seem to confirm the presence of two rather well-defined layers or regions of ionization.   The development of the theory has given an indication of what may be looked for and further experiments should be conducted with these things in mind.

While the reflection coefficients computed are, in fact, small (except quite near the critical frequency), and vary somewhat according to the law of variation of $n$ assumed, they apparently may reach several per cent on the basis of plausible assumption as to laws of variation of $n$, thickness of layer, wave length, etc., and it hence seems appropriate that this effect should be borne in mind as a possible explanation of some of the phenomena observed for, except very near the critical frequency, refraction can not explain the simultaneous appearance of rays from both layers unless horizontal gradients, that is, electron clouds, are assumed.   While such clouds are, of course, possible, it must be borne in mind that they would not be stable and hence would be expected to exist only under disturbed conditions.

With present high sensitivity radio receivers, a signal ratio of 100:1 corresponding to a reflection coefficient of 0.01 should, under favorable conditions, be inconsequential provided the general signal-to-noise level were sufficiently high (that is, high power transmission under favorable conditions).   Diurnal changes in sky wave energy of at least this order of magnitude are frequently observed on many frequencies.

WASHINGTON, August 21, 1931.